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Continuous time-interaction processes for population size estimation, with an application to drug dealing in Italy

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Appendix

Gradient of the conditional log-likelihood of M_b formulation

$$\frac{\partial l^*(\boldsymbol{\psi})}{\partial \alpha} = \sum_{i=1}^{n}\sum_{k=1}^{K_i}\left[\frac{\sum_{s<k} e^{-\beta(t_{ik}-t_{is})}}{\lambda(t_{ik})} + \frac{e^{-\beta(T-t_{ik})}-1}{\beta}\right]$$

$$\begin{aligned}\frac{\partial l^*(\boldsymbol{\psi})}{\partial \beta} &= \textstyle\sum_{i=1}^{n}\sum_{k=1}^{K_i}\left[\frac{\sum_{s<k}\alpha(t_{is}-t_{ik})e^{-\beta(t_{ik}-t_{is})}}{\lambda(t_{ik})} - \alpha(T-t_{ik})\frac{e^{-\beta(T-t_{ik})}}{\beta} - \alpha\left(\frac{e^{-\beta(T-t_{ik})}-1}{\beta^2}\right)\right] \\ &= \textstyle\alpha\sum_{i=1}^{n}\sum_{k=1}^{K_i}\left[\frac{\sum_{s<k}(t_{is}-t_{ik})e^{-\beta(t_{ik}-t_{is})}}{\lambda(t_{ik})} - \frac{1}{\beta^2}e^{-\beta(T-t_{ik})}\left[1+\beta(T-t_{ik})\right] + \frac{1}{\beta^2}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial l^*(\boldsymbol{\psi})}{\partial \eta} = &\textstyle\sum_{i=1}^{n}\left[\sum_{k=1}^{K_i}\frac{(\theta\eta t_{ik}+1)e^{-\theta(N_i(t_{ik})-\eta t_{ik})}}{\lambda(t_{ik})} - \sum_{k=0}^{K_i} e^{-\theta k}\left(t_{ik+1}e^{\theta\eta t_{ik+1}} - t_{ik}e^{\theta\eta t_{ik}}\right)\right] \\ &-nT\frac{\exp\left(\frac{1-e^{\theta\eta T}}{\theta}+\theta\eta T\right)}{1-\exp\left(\frac{1-e^{\theta\eta T}}{\theta}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\partial l^*(\boldsymbol{\psi})}{\partial \theta} = &\textstyle\sum_{i=1}^{n}\sum_{k=1}^{K_i}\frac{\eta(\eta t_{ik}-N_i(t_{ik}))e^{-\theta(N_i(t_{ik})-\eta t_{ik})}}{\lambda(t_{ik})} \\ &\textstyle+\frac{1}{\theta^2}\sum_{k=0}^{K_i} e^{-\theta k}\left[(1+\theta k-\theta\eta t_{ik+1})e^{\theta\eta t_{ik+1}} - (1+\theta k-\theta\eta t_{ik})e^{\theta\eta t_{ik}}\right] \\ &+\frac{n\exp\left(\frac{1-e^{\theta\eta T}}{\theta}\right)\left[e^{\theta\eta T}(1-\theta\eta T)-1\right]}{\theta^2\left(1-\exp\left(\frac{1-e^{\theta\eta T}}{\theta}\right)\right)}\end{aligned}$$

Gradient of the HT estimator for N in the M_b formulation

$$\frac{\partial \widehat{N}}{\partial \eta} = -\frac{nTe^{\frac{1-e^{\theta\eta T}}{\theta}+\theta\eta T}}{\left(1-e^{\frac{1-e^{\theta\eta T}}{\theta}}\right)^2}$$

$$\frac{\partial \widehat{N}}{\partial \theta} = -\frac{ne^{\frac{1-e^{\theta\eta T}}{\theta}}\left(-\frac{1-e^{\theta\eta T}}{\theta^2} - \frac{Te^{\theta\eta T+1}}{\theta}\right)}{\left(1-e^{\frac{1-e^{\theta\eta T}}{\theta}}\right)^2}$$

Gradient of the HT estimator for N in the M_{hotb} formulation

$$\frac{\partial P(K_i>0|c)^{-1}}{\partial \widetilde{\mu}_c} = -\frac{\exp\left\{\boldsymbol{X}_i'\boldsymbol{\gamma}+\widetilde{\mu}_c+\exp\left(\boldsymbol{X}_i'\boldsymbol{\gamma}+\widetilde{\mu}_c\right)\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt\right\}\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt}{\left\{1-\exp\left[\exp\left(\boldsymbol{X}_i'\boldsymbol{\gamma}+\widetilde{\mu}_c\right)\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt\right]\right\}^2}$$

$$\frac{\partial P(K_i > 0|c)^{-1}}{\partial \gamma_p} = -X_{ip}\frac{\exp\left\{\boldsymbol{X}_i'\boldsymbol{\gamma} + \widetilde{\mu}_c + \exp\left(\boldsymbol{X}_i'\boldsymbol{\gamma} + \widetilde{\mu}_c\right)\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt\right\}\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt}{\left\{1-\exp\left[\exp\left(\boldsymbol{X}_i'\boldsymbol{\gamma} + \widetilde{\mu}_c\right)\eta\int_0^T t^{\eta-1}e^{\widetilde{\theta}_c\eta t^{\eta-1}}dt\right]\right\}^2}$$

for $p = 1, \ldots, P$.